Limits at Infinity

When graphing a function, we are interested in what happens the values of the function as x becomes very large in absolute value. For example, if $f(x) = 1/x$ then as x becomes very large and positive, the values of $f(x)$ approach zero.

$$f(100) = \qquad f(1,000) = \qquad f(10,000) = \qquad f(1,000,000) =$$

$$f(-100) = \qquad f(-1,000) = \qquad f(-10,000) = \qquad f(-1,000,000) =$$

We say

$$\lim_{x\to\infty} 1/x = 0 \quad \text{and} \quad \lim_{x\to-\infty} 1/x = 0.$$

Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large or equivalently if for any number ϵ, there is a number M so that for all $x > M$, $|f(x) - L| < \epsilon$.

If f is defined on an interval $(-\infty, a)$, then we say

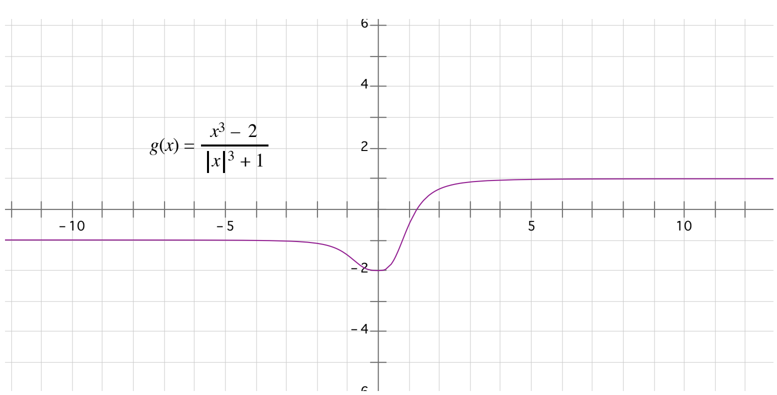
$$\lim_{x\to-\infty} f(x) = L$$

if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large and negative or equivalently
if for any number ϵ, there is a number N so that for all $x < N$, $|f(x) - L| < \epsilon$.

Note The symbol ∞ here does not represent a number, rather the symbol $\lim_{x\to\infty}$ means the limit as x becomes increasingly large.

Example Consider the graph of the function shown below. Judging from the graph, find are the limits

$$\lim_{x\to\infty} f(x) = \_\_\_\_\_\_\_\_ \qquad \lim_{x\to-\infty} f(x) = \_\_\_\_\_\_\_\_$$



We can see from the above graph that if $\lim_{x\to\infty} f(x) = L$, then the graph get closer and closer to the line $y = L$ as x approaches infinity.

Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either:

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Example What are the horizontal asymptotes of the graph of $y = \frac{x^3-2}{|x|^3+1}$ shown above?

We saw above that

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Example Find the following limits

$$\lim_{x\to\infty} \frac{2x+1}{x-5} \quad \text{and} \quad \lim_{x\to-\infty} \frac{2x+1}{x-5}.$$

Most of the usual limit laws hold for infinite limits with a replaced by ∞ or $-\infty$. The laws are listed below for reference :

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist (meaning they are finite numbers). Then

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
   (the limit of a sum is the sum of the limits).
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;
   (the limit of a difference is the difference of the limits).
3. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$;
   (the limit of a constant times a function is the constant times the limit of the function).
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$;
   (The limit of a product is the product of the limits).
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$;
   (the limit of a quotient is the quotient of the limits provided that the limit of the denominator is not 0)

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer (we see this using rule 4 repeatedly).
7. $\lim_{x\to a} c = c$, where c is a constant (easy to prove and easy to see from the graph, $y = c$).
8. $\lim_{x\to a} x = a$, (not difficult to prove from the definition and easy to see from the graph, $y = x$)
9. $\lim_{x\to\infty} \frac{1}{x} = 0$.
10. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ assuming that the $\lim_{x\to a} f(x) > 0$ if n is even.

Using 9, 6 and 10, we get;

Theorem If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Method For Rational Functions

We can use the above theorem to evaluate limits of rational functions at ∞ and $-\infty$. We divide both the numerator and denominator by the highest power of x in the denominator.

Example Evaluate

$$\lim_{x\to\infty} \frac{2x^2 + x + 1}{3x^2 - 1} \quad \text{and} \quad \lim_{x\to-\infty} \frac{2x^2 + x + 1}{3x^2 - 1}.$$

Find the vertical and horizontal asymptotes of the graph of

$$f(x) = \frac{2x^2 + x + 1}{3x^2 - 1}.$$

Definition Let f be a function defined on some interval (a, ∞). Then we say

$$\lim_{x\to\infty} f(x) = \infty$$

if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently large or equivalently if for any positive integer N, there is a number M so that for all $x > M$, $f(x) > N$.
We give similar meaning to the statements

$$\lim_{x\to\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = -\infty.$$

We have

$$\lim_{x\to\infty} x^n = \infty, \quad \lim_{x\to-\infty} x^{2n} = \infty \quad \lim_{x\to-\infty} x^{2n+1} = -\infty$$

for all positive integers n. Using this and law 10 above, we get that for all positive integers m, n

$$\lim_{x\to\infty} x^{\frac{n}{m}} = \infty, \quad \lim_{x\to-\infty} x^{\frac{2n}{2m+1}} = \infty \quad \lim_{x\to-\infty} x^{\frac{2n+1}{2m+1}} = -\infty$$

Example Evaluate

$$\lim_{x\to\infty} \frac{5x^3+x+1}{x^2-1}, \qquad \lim_{x\to-\infty} \frac{5x^3+x+1}{x^2-1}, \qquad \lim_{x\to\infty} \frac{5x+1}{x^2-4}, \qquad \lim_{x\to-\infty} \frac{5x+1}{x^2-4} \qquad \lim_{x\to-\infty} \frac{5x^5+1}{|x|^5-4}$$

Example Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+3}}{2x+5}, \qquad \lim_{x\to-\infty} \frac{\sqrt{3x^2+3}}{2x+5}, \qquad \lim_{x\to\infty} (\sqrt{x^2+x} - \sqrt{x^2-2x})$$

Note we can also use the squeeze theorem when calculating limits at ∞.

Example Find

$$\lim_{x\to\infty} \cos x, \quad \lim_{x\to\infty} \frac{\cos x}{x} \quad \lim_{x\to\infty} \sin\left(\frac{1}{x}\right), \quad \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right), \quad \lim_{x\to-\infty} \frac{5x+1}{x^2+\sin x - 4}.$$

if they exist.

Limits of Polynomials at Infinity and minus infinity

Let

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

be a polynomial function. Then the behavior of $P(x)$ at $\pm\infty$ is the same as that of its highest term. That is

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} a_n x^n \quad \text{and} \quad \lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} a_n x^n.$$

(To prove this consider the limit $\lim_{x\to\pm\infty} \frac{P(x)}{a_n x^n}$.)

Example Find

$$\lim_{x\to\infty} x^4 + 2x + 1, \quad \lim_{x\to-\infty} 2x^3 + x^2 + 1, \quad \lim_{x\to\infty} -3x^5 + 10x^2 + 4562x + 1, \quad \lim_{x\to\infty} (x-2)^3(x+1)^2(x-1)^5$$

Note that we can use the following short cut for calculating limits of rational functions as $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} \frac{ax^n + \text{lin. comb. of lower powers}}{bx^m + \text{lin. comb. of lower powers}} = \lim_{x\to\pm\infty} \frac{ax^n}{bx^m}$$

where m and n are positive integers.